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PROPAGATION OF HARMONIC THERMOELASTIC WAVES IN MEDIA WITH THERMAL MEMORY

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We present an analysis of the frequency dependence of phase velocities and damping coefficients of harmonic thermoelastic waves in media with thermal memory.

One of the urgent problems in heat-conduction theory is the investigation of heat-transfer processes taking thermal memory of the material into account [1]. By thermal memory here we mean the influence of the previous history of the thermal state of a body on its current state. Of special interest are the thermoelastic waves in such materials, which have propagation rates and damping coefficients different from the analogous relationships in the classical theory of thermoelasticity. Interest here is also stimulated by experiments on the propagation of thermal impulses at low temperatures [2] and in connection with high intensity thermal effects [3], where deviations from Hooke's Law are observed and where heat propagates at a finite rate in the form of waves of second sound.

A study of planar harmonic thermoelastic waves in the framework of a classical model was made by Chadwick [4]. Engel'brekht studied propagation modes of thermoelastic waves within a model of generalized thermomechanics [5] and within the Green–Lowe model [6].

We consider a one-dimensional mathematical model of linearized intercouple thermoelasticity for isotropic media, taking thermal memory into account [7]:

$$c_v\ddot{\vartheta}(z,\ t)+\tilde{\beta}(0)\dot{\vartheta}(z,\ t)+\int\limits_0^\infty\tilde{\beta}'(s)\dot{\vartheta}(z,\ t-s)\,ds=\tilde{\alpha}(0)\vartheta''(z,\ t)+\int\limits_0^\infty\tilde{\alpha}'(s)\vartheta''(z,\ t-s)\,ds+\varkappa_1\dot{u}'(z,\ t),$$
$$(2\varkappa_3+\varkappa_4)\,u''(z,\ t)-\rho\ddot{u}(z,\ t)=\varkappa_2\vartheta'(z,\ t)+\int\limits_0^\infty\tilde{\gamma}(s)\,\vartheta'(z,\ t-s)\,ds, \tag{1}$$

where a prime indicates differentiation with respect to the coordinate z; differentiation with respect to the time is indicated by an overdot.

We seek a solution of system (1) in the form of planar waves:

$$u=u_0\exp[i(\eta z-\omega t)],$$
$$\vartheta=\vartheta_0\exp[i(\eta z-\omega t)]. \tag{2}$$

Substituting relations (2) into Eqs. (1), we obtain the following characteristic equation:

$$(c_0^2\eta^2-\omega^2)\{c_v\omega^2+i\omega[\tilde{\beta}(0)+\tilde{\beta}_F'(\omega)]-\eta^2[\tilde{\alpha}(0)+\tilde{\alpha}_F'(\omega)]\}+\eta^2\omega^2\varkappa_1[\varkappa_2+\tilde{\gamma}_F(\omega)]=0, \tag{3}$$

where $f_F(\omega)$ denotes the Fourier transform of the relaxation function:

$$f_F(\omega)=\int\limits_0^\infty f(s)\exp(i\omega s)\,ds;\quad f=\{\tilde{\alpha}'(s),\ \tilde{\beta}'(s),\ \tilde{\gamma}(s)\}.$$

We write the characteristic equation (3) in dimensionless form:

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$$(\xi^2-\chi^2)\{n_1^*\chi^2+iN_*\chi[1+B(\chi)]-\xi^2[1+A(\chi)]\}+en_1^*\xi^2\chi^2[1+\Gamma_1\Gamma(\chi)]=0,$$

$$\chi=\omega/\omega_*,\quad \xi=\eta c_0/\omega_*,\quad \omega_*=c_0^2c_v/\lambda,$$

$$n_1^*=\lambda\omega_*/\tilde{\alpha}(0)=c_0^2/c_q^2,\quad c_q^2=\tilde{\alpha}(0)/c_v,$$

$$N_*=a\tilde{\beta}(0)/\tilde{\alpha}(0),\quad \Gamma_1=\tilde{\gamma}(0)\,a/\varkappa_2c_0^2,\quad e=\varkappa_1\varkappa_2/\rho c_vc_0^2,$$

$$A(\chi)=\int_0^\infty\alpha'(s)\exp(i\chi s)\,ds,\qquad \alpha'(s)=\tilde{\alpha}'(s)/\tilde{\alpha}(0), \tag{4}$$

$$B(\chi)=\int_0^\infty\beta'(s)\exp(i\chi s)\,ds,\quad \beta'(s)=\tilde{\beta}'(s)/\tilde{\beta}(0),$$

$$\Gamma(\chi)=\int_0^\infty\gamma(s)\exp(i\chi s)\,ds,\quad \gamma(s)=\tilde{\gamma}(s)/\tilde{\gamma}(0).$$

We shall assume that the relaxation functions have the form

$$\tilde{\alpha}(t)=\frac{\lambda}{\tau_q}\exp\left(-\frac{t}{\tau_q}\right),\qquad \tilde{\beta}(t)=\frac{c_v}{\tau_e}\exp\left(-\frac{t}{\tau_e}\right),$$
$$\tilde{\gamma}(t)=\frac{\varkappa_2}{\tau_\sigma}\exp\left(-\frac{t}{\tau_\sigma}\right). \tag{5}$$

Equation (4) can then be rewritten as follows:

$$\xi^4-\xi^2\left\{\chi^2[n_1(1+e)+1]+i\chi\left[1+e+\frac{1-i\chi n_1}{1-i\chi n_2}+e\frac{1-i\chi n_1}{1-i\chi n_3}\right]\right\}+i\chi^3+n_1\chi^4+i\chi^3\frac{1-i\chi n_1}{1-i\chi n_2}=0, \tag{6}$$
$$n_1=\omega_*\tau_q,\quad n_2=\omega_*\tau_e,\quad n_3=\omega_*\tau_\sigma.$$

We consider waves with fixed frequency, not taking into account thermal relaxation of stresses ($\tau_\sigma\gg\tau_q$, $\tau_\sigma\gg\tau_e$). Let the frequency ω be given by a real parameter. The solution is then obtained in the form of a sum of individual modes:

$$u=\exp(-i\chi\omega_*t)\sum_{k=1}^4a_k\exp\left(i\xi_k\omega_*\frac{z}{c_0}\right),$$
$$\vartheta=\exp(-i\chi\omega_*t)\sum_{k=1}^4b_k\exp\left(i\xi_k\omega_*\frac{z}{c_0}\right), \tag{7}$$

where a_k, b_k are amplitudes; ξ_k is determined from the solution of Eq. (6). It is convenient to represent the roots of equation (6) in the form

$$\xi_k=c_0(\chi/c_k-iq_k/\omega_*), \tag{8}$$

where c_k is the phase velocity ($c_k=c_0\chi(\mathrm{Re}\,\xi_k)^{-1}$); q_k is the damping coefficient ($q_k=q_0\,\mathrm{Im}\,\xi_k$, $q_0=\omega_*c_0^{-1}$).

We obtain asymptotic formulas for waves with fixed frequency. We denote the roots of Eq. (6) for e = 0 in the following way:

$$\xi_{1,3}^{(0)}=\pm\chi,\quad \xi_{2,4}^{(0)}=\pm\left(n_1\chi^2+i\chi\frac{2-i\chi(n_1+n_2)}{1-i\chi n_2}\right)^{1/2}, \tag{9}$$

where we have selected $\mathrm{Im}\,\xi_2^{(0)}>0$. Then for $\chi\ll1$ we obtain the following relations from Eqs. (9):

$$c_1=c_0,\qquad q_1=0; \tag{10}$$

$$\frac{c_2}{c_0}\approx\sqrt{\chi}\left(1-\frac{2n_1-n_2}{4}\chi+o(\chi^{5/2})\right); \tag{10a}$$

$$\frac{q_2}{q_0}\approx\sqrt{\chi}\left(1-\frac{2n_1-n_2}{2}\chi+o(\chi^{5/2})\right). \tag{10b}$$

The root $\xi_1^{(0)}$ corresponds to an elastic wave; the root $\xi_2^{(0)}$ corresponds to a thermal wave.

For large frequencies, for $\chi \gg 1$, we obtain the following from Eqs. (9):

$$c_1 = c_0, \qquad q_1 = 0; \tag{11a}$$

$$\frac{c_2}{c_0} \approx \frac{1}{\sqrt{n_1}} + o\left(\frac{1}{\chi^2}\right); \quad \frac{q_2}{q_0} \approx \frac{1}{2\sqrt{n_1}}\left(1 + \frac{n_1}{n_2}\right) + o\left(\frac{1}{\chi^3}\right); \tag{11b}$$

$$c_1^\infty = c_0, \quad c_2^\infty = c_q, \quad q_1^\infty = 0, \quad q_2^\infty = \frac{q_0}{2\sqrt{n_1}}\left(1 + \frac{n_1}{n_2}\right). \tag{11c}$$

For $e > 0$ we denote by ξ_1 that one of the roots of Eq. (5) which has for $\chi \ll 1$ and $e \to 0$ the asymptotic form (10), and we denote by ξ_2 that root which, under the same conditions, has the asymptotic form (10a), (10b). Then the wave corresponding to the root ξ_1 will behave for $\chi \ll 1$ as an elastic wave, while the wave corresponding to the root ξ_2 will behave as a thermal wave. We shall assume that ξ_k, $(k = 1, 2)$, are continuous and differentiable functions of χ. With these assumptions, for $e \neq 0$, $\chi \ll 1$ we expand the expression for ξ_k in powers of χ and we obtain the following asymptotic formulas:

$$\frac{c_1}{c_0} \approx \sqrt{\frac{2\chi}{m_0}}\left\{1 - \frac{D_1}{2m_0^2}\chi^2 + o(\chi^{7/2})\right\}; \tag{12}$$

$$\frac{q_1}{q_0} \approx \sqrt{\frac{\chi m_0}{2}}\left\{1 + \frac{D_2}{m_0}\chi^2 + o(\chi^{7/2})\right\}; \tag{12a}$$

$$\frac{c_2}{c_0} \approx \sqrt{\frac{m_0}{2}}\left\{1 - \frac{16\,[n_2^2(n_2 - n_1) - m_2] + m_3^2 m_0}{32 m_0} + o(\chi^3)\right\}; \tag{12b}$$

$$\frac{q_2}{q_0} \approx \frac{m_3\sqrt{m_0}}{4\sqrt{2}}\chi^2 + o(\chi^3); \tag{12c}$$

$$D_{1,2} = [m_1^2 + 4m_0(m_5 + m_3 \pm m_1)]/4m_0; \quad m_0 = 2 + e;$$

$$m_{1,4} = (2m_0 d_1 \pm k_0)/2m_0; \quad m_2 = F_0 m_0/4k_0; \quad m_3 = F_1/4m_0^3; \quad m_5 = (n_1 - n_2)/n_2^3;$$

$$F_0 = k_0\,[8m_0^4 k_1 - k_0^3 + 4m_0^2 k_0(n_0 - 2m_0 m_5)]/4m_0^6; \quad F_1 = k_0^2 - 4m_0^2(n_0 - 2m_0 m_5)$$

$$n_0 = [(2n_1 - n_2)^2 + 2(n_1 e - 1)(2n_1 - n_2) + (n_1 e + 1)^2];$$

$$k_0 = 2\,[m_0(2n_1 - n_2 + n_1 e) + e - 2];$$

$$k_1 = 2n_2(n_2 - n_1)(n_2 m_0 + 1 + n_2 - 2n_1 - n_1 e); \quad d_1 = 2n_1 - n_2 + 1 + n_1 e.$$

For large frequencies $\chi \gg 1$ we obtain

$$\frac{c_{1,2}}{c_0} \approx \frac{\sqrt{2}}{\sqrt{1 + n_1(1+e) \pm \sqrt{[1 + n_1(1+e)]^2 - 4n_1}}} + o\left(\frac{1}{\chi^2}\right); \tag{13}$$

$$\frac{q_{1,2}}{q_0} \approx \frac{c_{1,2}}{4c_0}\left\{1 + N + e \pm \frac{[1 + n_1(1+e)](1 + N + e) - 2(1+N)}{\sqrt{[1 + n_1(1+e)]^2 - 4n_1}}\right\} + o\left(\frac{1}{\chi^2}\right), \tag{13a}$$

where $N = n_1/n_2 = \tau_q/\tau_e$.

We consider now waves of fixed length. Let the wavelength be specified by a real parameter. Then the solution in the form of a sum of individual modes may be expressed as follows:

$$u = \exp\left(i\xi\omega_* \frac{z}{c_0}\right)\sum_{k=1}^{5} a_{1k}\exp(-i\omega_*\chi_k t),$$
$$\vartheta = \exp\left(i\xi\omega_* \frac{z}{c_0}\right)\sum_{k=1}^{5} b_{1k}\exp(-i\omega_*\chi_k t), \tag{14}$$

where a_{1k} and b_{1k} are amplitudes of the waves and χ_k is determined by solving the equation

$$\chi^5 + i\chi^4\left(\frac{2}{n_2} + \frac{1}{n_1}\right) - \chi^3\left[\frac{2}{n_1 n_2} + \xi^2\left(1 + e + \frac{1}{n_1}\right)\right] - \chi^2\xi^2 i\left[\frac{1}{n_2}\times\right.$$
$$\left.\times\left(2 + e + \frac{1}{n_1}\right) + \frac{1+e}{n_1}\right] + \chi\frac{\xi^2}{n_1}\left(\frac{2+e}{n_2} + \xi^2\right) + i\frac{\xi^4}{n_1 n_2} = 0. \tag{15}$$

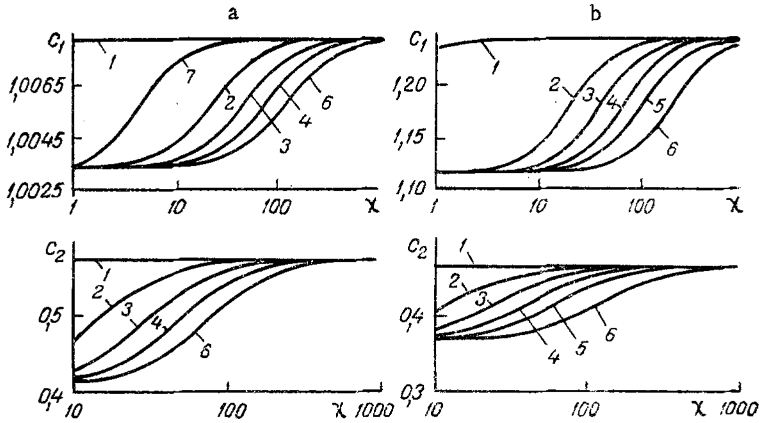

Fig. 1. Dependence of phase velocities $c_i = c_0\chi(\mathrm{Re}\,\xi_i)^{-1}$ on dimensionless frequency $\chi = \omega\omega_*^{-1}$: $n_1 = 3.0$; a) $e = 0.0114$; b) $e = 0.432$; Curve 1 corresponds to the model of generalized thermomechanics. Values of n_2 for curves 2, 3, 4, 5, 6, and 7 are, respectively, 0.1, 0.05, 0.03, 0.02, 0.01, and 0.5.

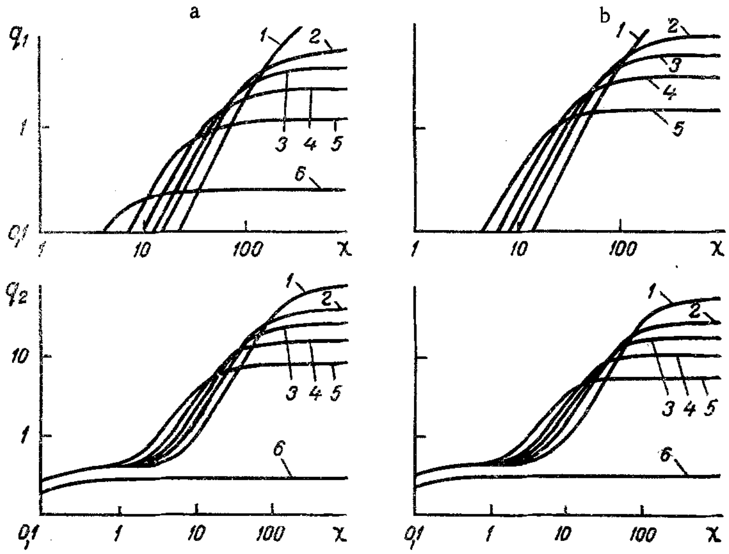

Fig. 2. Dependence of damping coefficients $q_i = q_0\,\mathrm{Im}\,\xi_i$, $q_0 = \omega_* c_0^{-1}$ on dimensionless frequency $\chi = \omega\omega_*^{-1}$: $n_1 = 3.0$; a) $e = 0.0114$; b) $e = 0.432$. Values of n_2 for curves 1, 2, 3, 4, and 5 are, respectively, 0.01, 0.02, 0.03, 0.05, and 0.1. Curve 6 corresponds to the model of generalized thermomechanics.

We consider modified waves without the quasithermal terms ($\vartheta^0 = 0$), propagating on the side on which the spatial coordinate increases. In this case

$$u = u_0 \exp[-\omega_* gt + \omega_*\xi c_0^{-1}(z - f\xi^{-1}c_0 t)],$$

$$\vartheta = u_0\varkappa_1 c_0 \frac{A}{\lambda}\exp[-\omega_* gt + \omega_*\xi c_0^{-1}(z - f\xi^{-1}c_0 t) + \pi + \delta], \tag{16}$$

where the roots of Eq. (15) are represented in the form $\chi = \pm f - ig$.

The connection between the wave deformation and the thermal wave stipulated by it is characterized by the coefficient A, which is given by the expression

$$A = \xi\sqrt{\frac{p_1^2 + f^2p_3^2}{(r_1 + p_1 - p_2\xi^2)^2 + f^2(p_3 - r_2 - n_2\xi^2)^2}}, \tag{17}$$

where

$$p_1 = r_1 + n_2(f^2 - g^2) + n_1 n_2 g(g^2 - 3f^2), \quad p_2 = 1 - n_2 g,$$
$$p_3 = n_1 n_2(f^2 - 3g^2) - r_2 + 2n_2 g, \quad r_1 = g + n_1(f^2 - g^2), \quad r_2 = 1 - 2n_1 g.$$

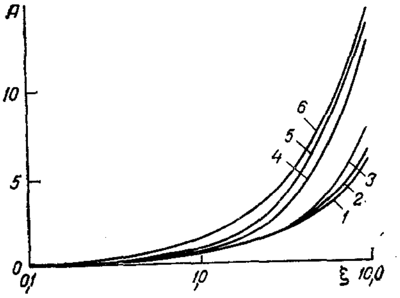

Fig. 3

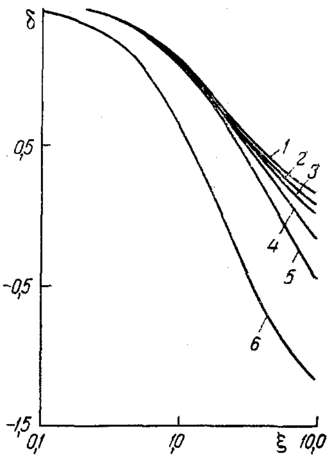

Fig. 4

Fig. 3. Dependence of coefficient A on dimensionless wave length $\xi = c_0\eta\omega_*^{-1}$; e = 0.0114; n_1 = 3.0. Values of n_2 for curves 1, 2, 3, 4, and 5 are, respectively, 0.01, 0.05, 0.1, 0.5, and 1.0. Curve 6 corresponds to the model of generalized thermomechanics.

Fig. 4. Dependence of the shear angle δ (rad) on the dimensionless wave length $\xi = c_0\eta\omega_*^{-1}$; e = 0.0114; n_1 = 0.5: Values of n_2 for curves 1, 2, 3, 4, and 5 are, respectively, 0.01, 0.02, 0.03, 0.05, and 0.1. Curve 6 corresponds to the model of generalized thermomechanics.

The thermal wave and the deformation wave are shifted in phase by the angle π + δ, where δ is determined from the relation

$$\tan\delta = \frac{f^2 p_3 (p_3 - r_2 - n_2\xi^2) + p_1 (r_1 + p_1 - p_2\xi^2)}{f\,[p_3 (r_1 - p_2\xi^2) + p_1 (r_2 + n_2\xi^2)]}. \tag{18}$$

For media with $\tau_e \gg \tau_q$ expressions (17) and (18) go over into well-known relations of generalized thermomechanics [5].

We consider now the behavior of waves with fixed frequency. Graphs showing the dependence of phase velocities and damping coefficients on frequency were obtained by numerically solving equation (6) using the binding coefficient for steel (e = 0.0114) and for polyvinylbutyral (e = 0.432).

Figure 1 shows the dependence of phase velocity on frequency for various values of the internal energy relaxation parameter n_2. Taking into account the additional relaxation mechanism leads to a decrease in velocity in comparison with the model of generalized thermomechanics. Influence of parameter n_2 is substantial in the frequency range $1.0 < \chi < 1000.0$. For large frequencies the phase velocity within the framework of a model which accounts for thermal memory ceases to depend on n_2 and tends towards the value of the velocity in the model of generalized thermomechanics. As the binding coefficient e increases, the speed of a fast wave increases while the speed of a slow wave decreases.

Figure 2 shows the dependence of damping coefficients on frequency for various values of parameter n_2. Taking into account the additional internal energy relaxation mechanism leads to an increase in the damping coefficient in comparison with the model of generalized thermomechanics. The most notable differences in the curves obtained when compared with the curves corresponding to the model of generalized thermomechanics may be seen in the range of frequencies $\chi > 1.0$. For large frequencies the damping coefficients cease to depend on the frequency and tend towards a finite limit. The limiting values of the damping coefficients decrease with an increase in the internal energy relaxation time. With an increase in the binding coefficient there is an increase in the damping of a fast wave and a decrease in the damping of a slow wave.

Figures 3 and 4 show the variation of the binding coefficient A and the shear angle δ as functions of the wave length ξ and the parameter n_2, which characterize waves with fixed length. Taking into account the additional internal energy relaxation mechanism leads to a decrease in the binding coefficient and an increase in the shear angle in comparison with the model of generalized thermomechanics. With a decrease in the internal energy relaxation time, the binding coefficient decreases and the shear angle increases. The difference in the binding coefficient may be observed for $\xi > 1.0$ while the difference in the shear angle may be observed for $\xi > 0.1$.

Thus, taking thermal memory into account is similar to taking relaxation of thermal flow into account [5] and taking rate of change of temperature into account [6], necessary for large frequencies or for small wavelengths. The relations we have obtained for velocities and damping of thermoelastic waves may find application in the experimental verification of thermoelastic models for the establishment of explicit expressions for heat flow and internal energy relaxation functions.

NOTATION

z, coordinate; t, time; ϑ, temperature; $\tilde{\alpha}(t)$, thermal flow relaxation function; $\tilde{\beta}(t)$, internal energy relaxation function; $\tilde{\gamma}(t)$, function of temperature relaxation of stresses; c_V, volumetric heat capacity; κ_i, linearization coefficients; ρ, density; u, displacement; ω, wave frequency; e, binding (compendency) coefficient; τ_q, thermal flow relaxation time; τ_e, internal energy relaxation time; τ_σ, time of temperature relaxation of stresses; λ, thermal conductivity; η, wavelength.

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SELF-SIMILAR SOLUTION OF THE PROBLEM OF CONSOLIDATION AND THAWING OF FROZEN SOIL

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The article presents a new mathematical model of the process of thawing of frozen soil taking consolidation into account. The following solutions were obtained: the self-similar one for the unidimensional biphase problem and an approximate analytical one for the simplified single-phase problem.

In the calculation of engineering structures erected in soil massif that thaws under their thermal effect, it is usual to take into account the thermal regime of the buildings and the position of the boundary between thawed and frozen zones of the accommodating soil in dependence on time [1-3]. Thus it is implicitly assumed that the heat source is the engineering structure and that it is fixed. In reality, however, there occurs filtering consolidation of the thawing soil; as a result the heat source moves according to the law of increasing subsidence [4] which takes into account the variable thickness of the consolidating soil layer.

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